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Ioffe's Normal Cone and the Foundations of Welfare Economics: An Example

M. Ali Khan

College of Commerce and Business Administration
Bureau of Economic and Business Research
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M. Ali Khan, Professor
Department of Economics

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by

M. Ali Khan*
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Abstract. We announce an example of an economy with an infinite dimensional commodity space for which the extension of the second fundamental theorem of welfare economics is valid if marginal rates of substitution are formalized in terms of either the Clarke normal cone or the Ioffe normal cone but in which the former is strictly contained in the latter. This is in direct contradiction to the finite dimensional situation.

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*Department of Economics, University of Illinois, 1206 South Sixth Street, Champaign, Illinois, 61820.

1. Introduction

In a recent paper, the author reported an extension of the so-called second fundamental theorem of welfare economics in which the marginal rates of substitution are formalized through the use of the Ioffe normal cone, see Khan [1987]. In particular, it was shown that under a mild constraint qualification and modulo scalar multiples, the Ioffe normal cones to the production and the "no-worse-than" sets at the respective production and consumption plans have a non-empty and non-zero intersection. If these sets are generated by differentiable functions, the Ioffe normal cones reduce to singletons and the result yields the conventional necessary conditions for a Pareto optimal allocation as in Hicks [1939], Lange [1942], Allais [1943], Samuelson [1947] and Graaf [1957]. If these sets are convex, the Ioffe normal cones reduce to the cones conventional in the sense of convex analysis and the result yields the statement that Pareto optimal allocations can be sustained through profit maximization by producers and expenditure minimization by consumers as in Arrow [1951], Debreu [1951, 1954], Malinvaud [1953] and Koopmans [1957]. Finally, since the Ioffe normal cone is contained in the Clarke normal cone, the result also generalizes recent work of Khan-Vohra [1987], Yun [1984], Quinzii [1986], Cornet [1986], which in turn generalized the earlier work of Guesnerie [1975].

The results in Khan [1987] are limited to an economy with a finite dimensional commodity space and were initially motivated by the consideration of an example of an economy with a production set without free disposal and for which the marginal rate of substitution at a

Pareto optimal production plan, as formalized by the Clarke normal cone at that plan, is the entire dual space. It was shown that the Ioffe normal cone at such a production plan is more in keeping with our intuitive notion of a marginal rate of substitution. In this note, we present an example of an economy with an infinite dimensional commodity space that furnishes an opposite conclusion. In particular, the economy has a production set without free disposal and for which the Ioffe normal cone at the Pareto optimal production plan is the entire dual space, but the Clarke normal cone is a strict subset. Intuition gets somewhat stretched in an infinite dimensional setting but the Clarke normal cone at such a production plan does not seem unreasonable. Our example underscores the unpredictability of the Ioffe normal cone in an infinite dimensional setting and is based on Treiman [1983].

Section 2 reviews some basic concepts and Section 3 presents the example. Section 4 is devoted to three concluding remarks.

2. The Basic Concepts

In this section we present the definitions of the Clarke and Ioffe cones. Since our example is set in an infinite dimensional space, we develop these definitions in the context of an arbitrary Banach space E. However, for concreteness, the reader may choose to think in terms of Euclidean n-space R^n or, for that matter, in terms of R^2.

We begin with a tangential approximant of a set introduced by Bouligand [1932] and termed the contingent cone. It was first applied in economics by Otani-Sicilian [1977].

<u>Definition 2.1</u> The <u>contingent cone</u> of $Y \subset E$ at $y \in Y$ is the set $T_K(Y,y) = \{x \in C: \exists$ a sequence $\{t^k\}$ of positive numbers with $t^k \to 0$ and a sequence $\{x^k\}$ with $x^k \to x$ such that $(y+t^k x^k) \in Y$ for all $k\}$.

<u>Definition 2.2</u> The <u>Clarke tangent cone</u> of $Y \subset E$ at $y \in Y$ is the set $T_C(Y,y) = \{x \in E$: For any sequence $\{t^k\}$ of positive numbers with $t^k \to 0$ and any sequence $\{y^k\}$ with $y^k \in Y$, $y^k \to y$, there exists a sequence $\{x^k\}$ with $x^k \to x$ such that $(y^k+t^k x^k) \in Y$ for all $k\}$.

<u>Definition 2.3</u> For any cone $A \subset E$, the <u>polar cone</u> A^+ is the set $\{y \in E^*: \langle y,x\rangle \leq 0$ for all $x \in A\}$ where E is the topological dual of E. [Note that $(R^n)^* = R^n$].

We shall denote the polars of $T_K(Y,y)$ and $T_C(Y,y)$ by $N_K(Y,y)$ and $N_C(Y,y)$ and refer to them as the <u>contingent normal cone</u> and <u>Clarke normal cone</u> respectively. For more details into these definitions, see Clarke [1983] and Khan-Vohra [1987].

<u>Definition 2.4</u> The <u>Ioffe normal cone</u> to $Y \subset E$ at $y \in Y$ is given by the set $N_a(Y,y) = \{x \in E: \exists$ a sequence $\{y^k\}$ with $y^k \in Y$, $y^k \to y$ and $x^k \in N_K(Y,y^k)$ with $x^k \to x\}$.

For details, see Ioffe [1981, 1984] and Khan [1987].

The reader can check his understanding of these basic concepts by referring to the technology depicted as Y in Figure 1a. The contingent cone $T_K(Y,y)$ is given by the shaded cone in Figure 1b, the Ioffe normal cone $N_a(Y,y)$ by the two arrows in Fig. 1b and the Clarke normal cone $N_C(Y,y)$ by the cone enclosed by these arrows. The Clarke

tangent cone $T_C(Y,y)$ is the set of all vectors making an obtuse angle with elements of $N_C(Y,y)$ and the contingent normal cone $N_K(Y,y)$ is zero.

We conclude this subsection by mentioning a result due to Cornet that brings out the central position of the contingent cone.

Theorem: Let Y be a nonempty closed set in R^n. Then $T_C(Y,y) = \{x \in R^n$: For all sequences $\{y^k\}$ with $y^k \in Y$, $y^k \to y$, there exists $x^k \in T_K(Y,y^k)$ with $x^k \to x\}$.

Proof: See Borwein-Strojwas [1985, Theorem 4.1].

3. The Example

We work in c_o, the space of sequences of real numbers converging to zero and endowed with the supremum norm. This is a Banach space with ℓ_1 as its dual (see, for example, Dunford-Schwartz [1957] for details and elementary properties). For any x,y in c_o, let $x \geq y$ denote $x_i \geq y_i$ for all coordinates i in N, the space of positive integers. Let $c_{o+} = \{x \in c_o: x_i \geq 0$ for all i in N$\}$. ℓ_{1+} has an analogous meaning.

The economy we present consists of c_o as the commodity space and two agents, a producer with production set $Y \subset c_o$ and a consumer with $0 \in c_o$ as the endowment, $X = \{x \in c_o: x_2 \geq 0, x_3 \geq 0\}$ as the consumption set and $\geq$ as the preference relation on X. The production set Y is given by

$$\{tw: t \geq 0\} \cup \{\bigcup_{m \in N} A_m\}$$

where

$$A_m = \{\frac{1}{m} z\} \cup \{\frac{1}{m} z + \bigcup_{n \in N} [\frac{1}{n}, \infty) \cdot \{\frac{1}{m^2} x_n + w\}\}$$

$$[a,b) \cdot A = \{ta: \quad t \in [a,b) \subset R \text{ and } a \in A\}$$

$$z = (1, -1, 0, 0, \ldots) \in c_o$$

$$w = (0, 0, 1, -1, 0, \ldots) \in c_o$$

$$x_n = - e_{n+4}, \; n \in N$$

and e_i is a vector in c_o with all coordinates zero except for the ith coordinate which is unity.

Note that Y is a technology capable of producing two outputs, the first and the third, and through the use of a countable number of inputs. More specifically, it consists of the technique w which can be operated at any nonzero level and the technique z which can be operated at any level $\frac{1}{m}$, m a positive integer. There is also the option of joint production of the first and third commodities but in this case "small" units of the third commodity require specialized inputs given by the elements of x_n.

<u>Lemma 3.1</u> Y is a non-empty, closed subset of c_o such that (i) $Y \cap c_{o+} = \{0\}$, (ii) $Y \cap (-Y) = \{0\}$, (iii) $T_K(Y, \frac{1}{m} z) = \{0\}$, (iv) $T_C(Y,0) = \{tw: \; t \geq 0\}$, (v) $N_a(Y,0) = \ell_1$.

Note that (i) formalizes the impossibility of producing something from nothing and (ii) the property of "irreversibility." Furthermore, Y is not convex and does not satisfy the property of "free disposal," i.e., for any $y \in Y$, $y - (c_{o+}) \not\subset Y$. For details of these properties, see the standard reference, Debreu [1959].

<u>Proof of Lemma 3.1</u> Nonemptiness of Y is is trivial and properties (i) and (ii) require routine computations. The shortest, though not necessarily the most transparent, proof of closedness, properties (iii) and (iv) can be had by the observation that Y satisfies all the assumptions required for Counterexample 3.1 in Treiman [1983]. (v) follows from Definition 2.4 and (iii).

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Now observe that $(0,0) \in (c_o \times c_o)$ is a Pareto optimal allocation of our two agent economy. If not, there exists $y \in Y$, $y \neq 0$, such that $y \geq 0$. But this contradicts Lemma 3.1(i). Now the "no worse than" set at 0 is given by c_{o+}. Furthermore, $N_C(c_{o+}, 0) = -\ell_{1+}$. Since $N_C(Y,0) = (T_C(Y,0))^+ = \{p \in \ell_1: \langle p,w\rangle \leq 0\}$, we obtain

$$[illegible] = N_C(Y,0) \; [illegible] \; (-N_C(c_{o+},0)) = \{p\in\ell_1: \; p_i \geq 0 \; (i\in N) \text{ and } p_3 \leq p_4\}$$

On the other hand,

$$[illegible] = N_a(Y,0) \; [illegible] \; (-N_a(c_{o+},0)) = \{p\in\ell_1: \; p_i \geq 0 \; (i\in N)\}.$$

The fact that [illegible], [illegible] $\neq$ [illegible] is forbidden by the principal result in the finite dimensional setup in Khan [1987].

4. Concluding Remarks

Ioffe [1981] presents an alternative definition of the so-called Ioffe normal cone that may lead to a different object in infinite dimensional spaces from the one considered here. For any set Y [illegible] E and $y \in Y$, denote such a cone by $N_a^I(Y,y)$ where

$$N_a^I(Y,y) = \bigcap_{F \in \mathcal{F}} \{x \in E: \exists \text{ a sequence } \{y^k\} \text{ with } y^k \to y,\ y^k \in Y$$

$$\text{and } x^k \in N_K(Y \cap (y^k+F), y^k) \text{ with } x^k \to x\},$$

and $\mathcal{F}$ is the family of finite dimensional subspaces of E. However, it is easy to check that for the example presented in Section 3,

$$N_a^I(Y,0) = \ell_1.$$

Borwein-Strojwas [1985] introduce the notion of <u>compactly epi-Lipschitzian</u> sets that includes the class of epi-Lipschitzian sets introduced in the context of the second welfare theorem by Khan-Vohra [1985]. It is easy to check that Y in the example in Section 3 is not compactly epi-Lipschitzian at the Pareto-optimal production plan consisting of zero. One simply needs the characterization of compact sets in c_o as given in Dunford-Schwartz [1957, IV.13.9].

Finally, it is easy to manufacture examples of economies with an infinite dimensional commodity space in which the extension of the second welfare theorem fails in the sense that the normal cones, Clarke's or Ioffe's, at the Pareto optimal production and consumption plans have an intersection consisting solely of the zero vector. For one such example, simply take the negative of set presented in Klee [1963] as the production set plus initial endowment and the coordinatewise ordering as the preference relation. The example presented in Section 3 is very different in spirit. Here the problem has to do with the Ioffe normal cone being "too large" rather than "too small."

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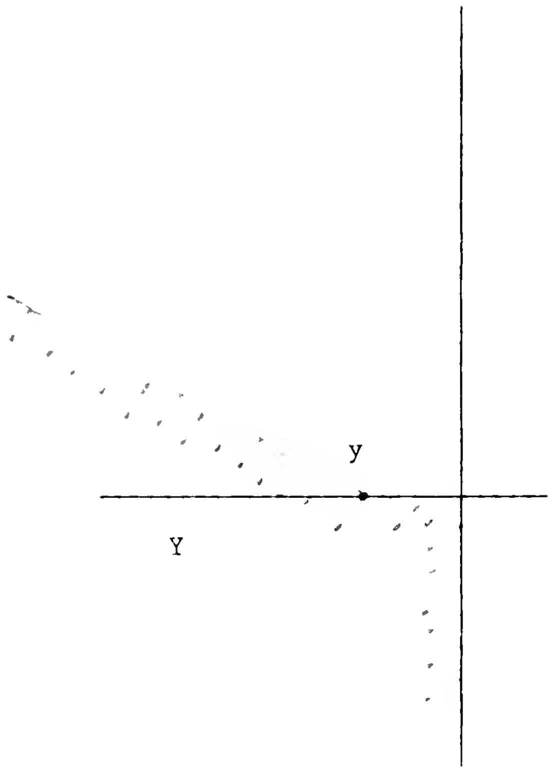

Fig. 1a

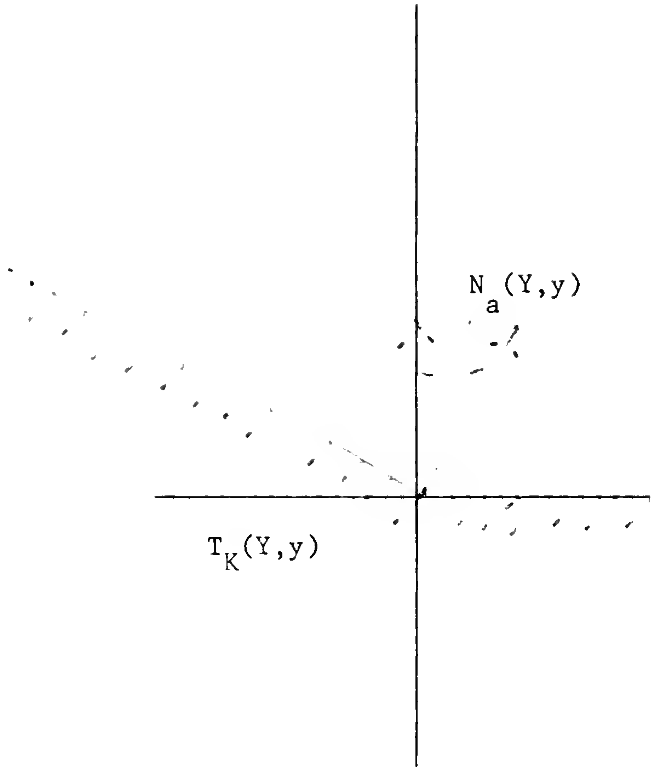

Fig 1b